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Transport of energy by disturbances in arbitrary steady flows

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An exact equation governing the transport of energy associated with disturbances in an arbitrary steady flow is derived. The result is a generalization of the familiar concept of acoustic energy and is suggested by a perturbation expansion of the general energy equation of fluid mechanics. A disturbance energy density and flux are defined and identified as exact fluid dynamic quantities whose leading-order regular perturbation representations reduce in various special cases to previously known results. The exact equation on disturbance energy is applied to a simple example of nonlinear wave propagation as an illustration of its general utility in situations where a linear description of the disturbance is inadequate.

1. Introduction

Suitable measures of the energy associated with disturbances to a given flow have been the subject of considerable interest and debate for many years. Expressions for energy are necessarily of second degree in the dependent variables which describe a disturbance, but if that disturbance is treated according to a first-order approximate theory, for example, then any useful measure of energy must contain only first-order approximations as well. Particularly in linear acoustics, questions about the order consistency of the commonly defined acoustic energy density and flux and their relationship to the general principle of energy conservation continue to arise. Such considerations are of obvious theoretical interest; in acoustics they stem also from a practical need to explore the possibility of making direct measurements of acoustic power transmission in complicated background flows. As indicated, the concepts to be discussed here have perhaps received most attention in the study of sound propagation, and it was in this context that the author originally approached the subject. Thus, some of the following is presented from an acoustic point of view. It is emphasized, however, that the results to be developed have application to any fluid mechanical situation in which attention is directed to the behaviour of disturbances propagating in a known basic flow. Hence, with appropriate specialization of fluid properties and flow characteristics, and proper choice of frame of reference, they apply to problems of steady and unsteady aerodynamics, stability, turbulence, and so on.

Although no detailed review of previous work on the subject will be attempted here, there are certain sources which bear directly on the present paper. Perhaps the best known early discussion in the acoustic literature of energy in moving media is that of Cantrell & Hart (1964), who considered homentropic, irrotational flow. Previously, Blokhintsev (1956) developed similar ideas in the high-frequency context of geometric acoustic theory. The work of Morfey (1971) seems to be the first comprehensive analysis of the energetics of disturbances to general fluid flows, and included there is an extensive bibliography of prior discussions of acoustic energy relations. Of course, the fundamental significance of acoustic energy in a quiescent uniform medium has been appreciated since the time of Rayleigh; a clear modern exposition is given by Lighthill (1978), who also points out the unnacceptibility of basing the definition of acoustic energy on time-averaging of fluctuating quantities. Nevertheless, it is an indication of the troublesome nature of the subject that debate can still be generated about even the simplest classical case (Chu & Apfel 1983). An important source, which is also a valuable guide to the earlier literature, is the textbook by Pierce (1981). Pierce develops, both in the text and in the exercises, a sequence of energy corollaries applicable to increasingly more general mean states.

The point of view taken in the present discussion differs from that represented by the sources cited above. They proceed, as do virtually all similar analyses known to the author, from the premise that disturbances to a flow are of sufficiently small magnitude and are otherwise suitably well behaved that they can be represented by a regular perturbation expansion in which the leading approximation is described by a linearized theory. In any physical problem, however, such a description is likely not to be uniformly valid. As an example, one can point to the fact that propagation of sound through a flow region where the Mach number is near unity in an inherently nonlinear phenomenon and that linear acoustic theory is a singular perturbation in this circumstance. A convenient means of accounting for the transport of energy through such a region can often be of great value in view of the fact that losses occur because of the development of shocks in the perturbation field. This cannot be accomplished, however, using linear measures of energy density and flux, which are meaningless. As a result, another fundamental question arises which is, in essence, the topic of the current paper. Specifically, it becomes of major interest to determine precisely what *exact* fluid dynamic quantities are represented in the linear approximation by various commonly accepted expressions for energy density and flux. In the following, an answer to this question is deduced through the derivation of an *exact* energy corollary applicable to arbitrary disturbances in an arbitrary steady flow. The corollary is constructed so as to be a complete, consistent representation of the principle of conservation of total fluid energy and so that it has a leading-order representation in a regular perturbation scheme which is identical, in the relevant special cases, to those mentioned above. Because it is completely general, however, it is a straightforward matter to determine the form it assumes under any approximation scheme appropriate to a given physical problem. The author has given greatly restricted versions of the present analysis previously in studies of sound transmitted through a near-sonic duct throat (Myers 1981; Myers & Callegari 1982) and for homentropic ideal flow (Myers 1986*b*), and a preliminary form of some of the following was laid out in Myers (1986*a*). A simple nonlinear problem is treated briefly in the last section of the present paper as an application of the general result developed.

2. Formulation

The set of equations governing the motion of a real fluid in the absence of body forces can be written in Cartesian coordinates as

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_i}=0, \tag{1a}$$

$$\frac{\partial u_j}{\partial t}+u_i\frac{\partial u_j}{\partial x_i}+\frac{1}{\rho}\frac{\partial p}{\partial x_j}=\frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i}, \tag{1b}$$

$$\frac{\partial s}{\partial t}+u_i\frac{\partial s}{\partial x_i}=\frac{1}{\rho T}\left(\Phi-\frac{\partial q_i}{\partial x_i}\right), \tag{1c}$$

in which ρ is the density, the u_j are the components of the velocity vector $\boldsymbol{u}$, p is the thermodynamic pressure and s is the specific entropy. On the right-hand side of (1), P_{ij} denotes the components of the viscous stress tensor, Φ is the dissipation function $P_{ij}\,\partial u_j/\partial x_i$, T is the absolute temperature and the q_i are the components of the heat flux vector. In addition, the present analysis will utilize three well-known thermodynamic relations (see e.g. Landau & Lifshitz 1959):

$$\mathrm{d}e=T\,\mathrm{d}s+\frac{p}{\rho^2}\mathrm{d}\rho,\quad \mathrm{d}p=\frac{c^2\rho\beta T}{c_p}\mathrm{d}s+c^2\,\mathrm{d}\rho,\quad \mathrm{d}T=\frac{T}{c_p}\mathrm{d}s+\frac{\beta T}{\rho c_p}\mathrm{d}p. \tag{2a–c}$$

Here e is the specific internal energy, c is the speed of sound, c_p is the specific heat at constant pressure and β is the coefficient of thermal expansion, $(-\partial\rho/\rho\,\partial T)_p$.

For purposes of algebraic simplicity in the following, it is convenient to rewrite the system (1), expressing it in vector notation. Use relation (2a) to write $(\boldsymbol{\nabla}p)/\rho=\boldsymbol{\nabla}h-T\boldsymbol{\nabla}s$, where h is the specific enthalpy $e+p/\rho$. Upon use of a familiar vector identity, the convective acceleration in (1b) can be written $\boldsymbol{\nabla}(\frac{1}{2}u^2)+\boldsymbol{\xi}\times\boldsymbol{u}$, where $\boldsymbol{\xi}$ is the vorticity $\boldsymbol{\nabla}\times\boldsymbol{u}$. Now define vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\psi}$ such that $\boldsymbol{\zeta}=\boldsymbol{\xi}\times\boldsymbol{u}$ and $\psi_j=(1/\rho)(\partial P_{ij}/\partial x_i)$, and let $(\Phi-\boldsymbol{\nabla}\cdot\boldsymbol{q})/T=Q$. Then (1) assumes the form

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{m}=0, \tag{3a}$$

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{\zeta}+\boldsymbol{\nabla}H-T\boldsymbol{\nabla}s=\boldsymbol{\psi}, \tag{3b}$$

$$\frac{\partial(\rho s)}{\partial t}+\boldsymbol{\nabla}\cdot(\boldsymbol{m}s)=Q. \tag{3c}$$

In (3), H is the specific stagnation enthalpy $h+\frac{1}{2}u^2$, $\boldsymbol{m}$ is the mass flux vector $\rho\boldsymbol{u}$ and (3c) has been obtained by multiplying (1c) by ρ and using (1a). The system (3) will be taken as the fundamental working equations in what follows. It can be completed by specification of state equations giving h and T as functions of ρ and s.

For reasons that will become clear later, some of the present analysis will be carried out quite formally by representing each fluid quantity $q(\boldsymbol{x},t)$ in the form

$$q(\boldsymbol{x},t)=q_0(\boldsymbol{x})+\sum_{n=1}^{\infty}\delta^n q_n(\boldsymbol{x},t), \tag{4}$$

in which δ is a small parameter which measures the order of magnitude of unsteady disturbances to a basic steady flow $q_0(\boldsymbol{x})$. Substituting the expansions (4) into (3) and equating coefficients of like powers of δ to zero independently then leads to a sequence of systems of equations which govern the nth-order fluid quantities. The first three of these are

$$\nabla\cdot\boldsymbol{m}_0 = 0, \tag{5a}$$

$$\boldsymbol{\zeta}_0+\nabla H_0-T_0\nabla s_0 = \boldsymbol{\psi}_0, \tag{5b}$$

$$\nabla\cdot(\boldsymbol{m}_0 s_0) = Q_0; \tag{5c}$$

$$\frac{\partial\rho_1}{\partial t}+\nabla\cdot\boldsymbol{m}_1 = 0, \tag{6a}$$

$$\frac{\partial\boldsymbol{u}_1}{\partial t}+\boldsymbol{\zeta}_1+\nabla H_1-T_0\nabla s_1-T_1\nabla s_0 = \boldsymbol{\psi}_1, \tag{6b}$$

$$\frac{\partial}{\partial t}(\rho_0 s_1+\rho_1 s_0)+\nabla\cdot(\boldsymbol{m}_0 s_1+\boldsymbol{m}_1 s_0) = Q_1; \tag{6c}$$

$$\frac{\partial\rho_2}{\partial t}+\nabla\cdot\boldsymbol{m}_2 = 0, \tag{7a}$$

$$\frac{\partial\boldsymbol{u}_2}{\partial t}+\boldsymbol{\zeta}_2+\nabla H_2-T_0\nabla s_2-T_1\nabla s_1-T_2\nabla s_0 = \boldsymbol{\psi}_2, \tag{7b}$$

$$\frac{\partial}{\partial t}(\rho_0 s_2+\rho_1 s_1+\rho_2 s_0)+\nabla\cdot(\boldsymbol{m}_0 s_2+\boldsymbol{m}_1 s_1+\boldsymbol{m}_2 s_0) = Q_2. \tag{7c}$$

Now, to simpify writing some of the relations which occur later it is useful to define the left-hand sides of the continuity, linear momentum and entropy equations in (3) as C, $\boldsymbol{L}$ and S, respectively, with corresponding definitions C_n, $\boldsymbol{L}_n$ and S_n used for (5), (6) and (7). Thus (3) is abbreviated as

$$C = 0, \quad \boldsymbol{L}-\boldsymbol{\psi} = 0, \quad S-Q = 0, \tag{8a–c}$$

and each of (5), (6) and (7) is

$$C_n = 0, \quad \boldsymbol{L}_n-\boldsymbol{\psi}_n = 0, \quad S_n-Q_n = 0, \tag{9a–c}$$

for $n = 0, 1, 2$.

To complete the nth-order systems above requires corresponding expansions of the thermodynamic quantities h and T. Actually, the explicit form of T is not required in the following, but it will be necessary to utilize expansions of ρe and p. These can all be derived by first expanding each as power series in $\rho-\rho_0$ and $s-s_0$. For example,

$$\rho e = \rho_0 e_0+\left.\frac{\partial(\rho e)}{\partial\rho}\right|_0(\rho-\rho_0)+\left.\frac{\partial(\rho e)}{\partial s}\right|_0(s-s_0)+\left.\frac{\partial^2(\rho e)}{\partial\rho^2}\right|_0\frac{(\rho-\rho_0)^2}{2} +\left.\frac{\partial^2(\rho e)}{\partial\rho\,\partial s}\right|_0(\rho-\rho_0)(s-s_0)+\left.\frac{\partial^2(\rho e)}{\partial s^2}\right|_0\frac{(s-s_0)^2}{2}+\dots. \tag{10}$$

Use of (2) allows the required derivatives in (10) to be evaluated. Thus,

$$\left.\begin{aligned}&\frac{\partial(\rho e)}{\partial\rho} = h; \quad \frac{\partial(\rho e)}{\partial s} = \rho T; \quad \frac{\partial^2(\rho e)}{\partial\rho^2} = \frac{\partial h}{\partial\rho} = \frac{c^2}{\rho};\\ &\frac{\partial^2(\rho e)}{\partial s^2} = \frac{\rho T}{c_p}\left(1+\frac{c^2\beta^2 T}{c_p}\right); \quad \frac{\partial^2(\rho e)}{\partial\rho\,\partial s} = \frac{\partial h}{\partial s} = T\left(1+\frac{c^2\beta}{c_p}\right).\end{aligned}\right\} \tag{11}$$

Then, after substitution of the expansions (4) for ρ and s, equation (10) becomes

$$\rho e = \rho_0 e_0+\delta(h_0\rho_1+\rho_0 T_0 s_1) +\delta^2\left(h_0\rho_2+\rho_0 T_0 s_2+\rho_1 T_0 s_1+\frac{p_1^2}{2\rho_0 c_0^2}+\frac{\rho_0 T_0 s_1^2}{2c_{p_0}}\right)+O(\delta^3). \tag{12}$$

The same procedure yields h and p in the form

$$h = h_0 + \delta\left(\frac{p_1}{\rho_0} + T_0 s_1\right) + O(\delta^2), \tag{13}$$

$$p = p_0 + \delta\left(c_0^2 \rho_1 + \frac{\rho_0 c_0^2 \beta_0 T_0 s_1}{c_{p_0}}\right) + O(\delta^2). \tag{14}$$

In deriving the expression for $(\rho e)_2$ in (12) and for h_1 in (13) the expression for p_1 in terms of ρ_1 and s_1 from (14) has been used. In each of these expansions the subscript 0 in the coefficients indicates values in the unperturbed state $q_0(\boldsymbol{x})$.

It is appropriate to note before going further that the above process of successive approximation is not new and has been utilized, sometimes less formally, by many earlier authors to arrive at various forms of the system (7). That system governs second-order corrections to the linearized theory embodied in the system (6). In acoustics, for example, such corrections have been the subject of many important publications during the past several decades which form the basis of our understanding of phenomena such as acoustic radiation pressure and steady streaming (see e.g. Eckart 1948). Although many are not directly relevant to the present analysis, some have treated higher-order acoustic behaviour in terms of energy considerations. Readers new to these topics may wish to consult the many sources cited in Morfey (1971) and Pierce (1981).

Now, the system of equations (6) along with the expression for h_1 from (13) and a corresponding expression for T_1 as a function of ρ_1 and s_1 constitute a complete formulation of the linearized theory of small perturbations around a general steady flow satisfying the system (5). It is seen that by developing the theory in the above manner no explicit use has been made of the general equation which expresses the principle of conservation of total fluid energy; it has been replaced by its equivalent in terms of entropy, equation (1*c*). It is well known, however, that the system (6) can be manipulated algebraically in numerous special cases to derive relations which apparently express some version of the energy principle. For example, for homentropic irrotational flow of a perfect fluid one can show that (6) lead to an energy corollary of the form

$$\frac{\partial E_a}{\partial t} + \nabla \cdot \boldsymbol{W}_a = 0 \tag{15}$$

in which the energy density E_a is given by

$$E_a = \frac{p_1^2}{2\rho_0 c_0^2} + \frac{\rho_0 u_1^2}{2} + \rho_1 \boldsymbol{u}_0 \cdot \boldsymbol{u}_1, \tag{16}$$

and the energy flux vector $\boldsymbol{W}_a$ is

$$\boldsymbol{W}_a = (p_1 + \rho_0 \boldsymbol{u}_0 \cdot \boldsymbol{u}_1)\left(\boldsymbol{u}_1 + \frac{\rho_1}{\rho_0}\boldsymbol{u}_0\right). \tag{17}$$

$\boldsymbol{W}_a$ in (17) is the 'acoustic' energy flux first defined (in time-averaged form with $\rho_1 = p_1/c_0^2$) by Cantrell & Hart (1964). The utility of a result such as (15) lies in the fact that it expresses what is fundamentally a second-order balance of energy but it involves only first-order perturbation quantities.

As mentioned earlier, relations such as (15) have been derived and debated in many different contexts for some time. Of particular concern when they are produced simply by algebraic manipulation of the linear system (6) is the question of how they relate to the general energy principle and to what order of approximation they represent it. Of course, for the special case cited above, if $\boldsymbol{u}_0 = 0$ and if the other zeroth-order quantities are constants, there is no real difficulty (see e.g. Lighthill 1978). The picture becomes much more clouded, however, when the basic state involves a fluid in motion. In the following section, a first-order result corresponding to (15) will be deduced directly from a formal expansion of the general energy equation for the fluid. While this approach is somewhat more cumbersome than the customary procedure of algebraic combination of the equations of the linear system (6), it does lead to a very general result which does not appear to have been given elsewhere. The primary motivation for the analysis, however, is that it provides a clear demonstration of the relation between energy corollaries such as (15) and the general energy equation and of their order consistency with it. Just as important is the fact that the specific details of the analysis suggest the generalization which is the major result of the current paper.

3. First-order energy corollary

The general energy equation, which reflects the fact that total fluid energy (internal plus kinetic) is conserved is expressed, in the absence of body forces, in the form (Landau & Lifshitz 1959; Batchelor 1967)

$$\frac{\partial}{\partial t}(\rho H - p) + \frac{\partial}{\partial x_i}(m_i H + q_i - P_{ij} u_j) = 0, \tag{18a}$$

or, in the previously defined notation,

$$\frac{\partial}{\partial t}(\rho H - p) + \nabla\cdot(\boldsymbol{m}H) - \boldsymbol{m}\cdot\boldsymbol{\psi} - TQ = 0. \tag{18b}$$

From the point of view of continuum fluid mechanics, this equation is the fundamental statement about energy conservation in a flow. As is well known, the entropy equation (1c) is a consequence of (18). Here it will be considered the other way around: given the set of equations (1), then (18) is not independent but can be treated as a corollary to that set. Any solution to the complete set (1) automatically satisfies (18). From this viewpoint it is of interest to examine the consequences of introducing the formal expansions (4) into (18). Substituting the expansions into the energy equation and equating coefficients of like powers of δ to zero independently leads to a sequence of equations which express conservation of total energy at the nth order. For $n = 0$, 1, 2 these are

$$\nabla\cdot(\boldsymbol{m}_0 H_0) - \boldsymbol{m}_0\cdot\boldsymbol{\psi}_0 - T_0 Q_0 = 0, \tag{19a}$$

$$\frac{\partial}{\partial t}(\rho H - p)_1 + \nabla\cdot(\boldsymbol{m}_0 H_1 + \boldsymbol{m}_1 H_0) - \boldsymbol{m}_0\cdot\boldsymbol{\psi}_1 - \boldsymbol{m}_1\cdot\boldsymbol{\psi}_0 - T_0 Q_1 - T_1 Q_0 = 0, \tag{19b}$$

$$\frac{\partial}{\partial t}(\rho H - p)_2 + \nabla\cdot(\boldsymbol{m}_0 H_1 + \boldsymbol{m}_1 H_1 + \boldsymbol{m}_2 H_0) - \boldsymbol{m}_0\cdot\boldsymbol{\psi}_2 - \boldsymbol{m}_1\cdot\boldsymbol{\psi}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\psi}_0 - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0. \tag{19c}$$

Each of equations (19) can be viewed as being a corollary to the corresponding nth-order systems (5), (6) and (7). What they actually express will be analysed in detail in the following paragraphs.

Consider (19*a*); it is equivalent to

$$H_0 \nabla\cdot \boldsymbol{m}_0 + \boldsymbol{m}_0\cdot(\nabla H_0 - \boldsymbol{\psi}_0) - T_0 Q_0 = 0,$$

or
$$(H_0 - T_0 s_0)\nabla\cdot\boldsymbol{m}_0 + \boldsymbol{m}_0\cdot(\boldsymbol{L}_0 + T_0\nabla s_0 - \boldsymbol{\zeta}_0 - \boldsymbol{\psi}_0) + T_0 s_0 \nabla\cdot\boldsymbol{m}_0 - T_0 Q_0 = 0. \tag{20}$$

Because $\boldsymbol{m}_0\cdot\boldsymbol{\zeta}_0 = \rho_0\boldsymbol{u}_0\cdot(\boldsymbol{\xi}_0\times\boldsymbol{u}_0) \equiv 0$, equation (20) is

$$(H_0 - T_0 s_0) C_0 + \boldsymbol{m}_0\cdot(\boldsymbol{L}_0 - \boldsymbol{\psi}_0) + T_0(S_0 - Q_0) = 0, \tag{21}$$

in which the abbreviations defined in (9) have been introduced. In view of (9), it is seen that the energy equation at order 0 contains no new information; given any solution of the system (5), (19*a*) is simply an identity.

Equation (19*b*) can be handled similarly. First, note that

$$(\rho H - p)_1 = (\rho e)_1 + \tfrac{1}{2}\rho_1 u_0^2 + \rho_0\boldsymbol{u}_0\cdot\boldsymbol{u}_1 = H_0\rho_1 + \rho_0 T_0 s_1 + \boldsymbol{m}_0\cdot\boldsymbol{u}_1 \tag{22}$$

after (12) has been used. Then (19*b*) becomes, utilizing (5*a*) and (5*b*),

$$H_0\left(\frac{\partial\rho_1}{\partial t} + \nabla\cdot\boldsymbol{m}_1\right) + T_0\frac{\partial}{\partial t}(\rho_0 s_1) + \boldsymbol{m}_0\cdot\left(\frac{\partial\boldsymbol{u}_1}{\partial t} + \nabla H_1 - \boldsymbol{\psi}_1\right)$$
$$+ \boldsymbol{m}_1\cdot(T_0\nabla s_0 - \boldsymbol{\zeta}_0) - T_0 Q_1 - T_1 Q_0 = 0,$$

or

$$(H_0 - T_0 s_0) C_1 + T_0\frac{\partial}{\partial t}(\rho_0 s_1 + \rho_1 s_0) + T_0 s_0\nabla\cdot\boldsymbol{m}_1$$
$$+ \boldsymbol{m}_0\cdot(\boldsymbol{L}_1 + T_0\nabla s_1 + T_1\nabla s_0 - \boldsymbol{\zeta}_1 - \boldsymbol{\psi}_1) + \boldsymbol{m}_1\cdot(T_0\nabla s_0 - \boldsymbol{\zeta}_0) - T_0 Q_1 - T_1 Q_0 = 0. \tag{23}$$

Upon taking account of (5*c*) this is

$$(H_0 - T_0 s_0) C_1 + \boldsymbol{m}_0\cdot(\boldsymbol{L}_1 - \boldsymbol{\psi}_1) + T_0(S_1 - Q_1) - \boldsymbol{m}_0\cdot\boldsymbol{\zeta}_1 - \boldsymbol{m}_1\cdot\boldsymbol{\zeta}_0 = 0. \tag{24}$$

Now, because $\boldsymbol{m}_1 = \rho_0\boldsymbol{u}_1 + \rho_1\boldsymbol{u}_0$ and $\boldsymbol{\zeta}_1 = \boldsymbol{\xi}_0\times\boldsymbol{u}_1 + \boldsymbol{\xi}_1\times\boldsymbol{u}_0$, the vorticity terms in (24) can be shown to reduce to $-\rho_0\boldsymbol{u}_0\cdot(\boldsymbol{\xi}_0\times\boldsymbol{u}_1) - \rho_0\boldsymbol{u}_1\cdot(\boldsymbol{\xi}_0\times\boldsymbol{u}_0) \equiv 0$. Therefore, it follows again, when (9) are used, that the energy equation (19*b*) is simply an identity for any solution of the system (6). It is noted that this occurs, as it did for (19*a*), regardless of the presence of entropy and vorticity variations in the solutions of the systems (5) and (6).

Finally, consider (19*c*). From (12) it follows that

$$(\rho H - p)_2 = (\rho e)_2 + \tfrac{1}{2}\rho_0 u_1^2 + \rho_1\boldsymbol{u}_0\cdot\boldsymbol{u}_1 + \tfrac{1}{2}\rho_2 u_0^2 + \rho_0\boldsymbol{u}_0\cdot\boldsymbol{u}_2$$
$$= \frac{p_1^2}{2\rho_0 c_0^2} + \tfrac{1}{2}\rho_0 u_1^2 + \rho_1\boldsymbol{u}_0\cdot\boldsymbol{u}_1 + \frac{\rho_0 T_0 s_1^2}{2c_{p_0}} + H_0\rho_2 + T_0(\rho_0 s_2 + \rho_1 s_1) + \boldsymbol{m}_0\cdot\boldsymbol{u}_2. \tag{25}$$

Introduce the definition

$$E_2 = \frac{p_1^2}{2\rho_0 c_0^2} + \tfrac{1}{2}\rho_0 u_1^2 + \rho_1\boldsymbol{u}_0\cdot\boldsymbol{u}_1 + \frac{\rho_0 T_0 s_1^2}{2c_{p_0}}, \tag{26}$$

so that, after utilizing (5*a*), (5*b*) and (25), equation (19*c*) becomes

$$\frac{\partial E_2}{\partial t} + H_0\left(\frac{\partial\rho_2}{\partial t} + \nabla\cdot\boldsymbol{m}_2\right) + T_0\frac{\partial}{\partial t}(\rho_0 s_2 + \rho_1 s_1) + \boldsymbol{m}_0\cdot\left(\frac{\partial\boldsymbol{u}_2}{\partial t} + \nabla H_2 - \boldsymbol{\psi}_2\right)$$
$$+ \nabla\cdot(\boldsymbol{m}_1 H_1) - \boldsymbol{m}_1\cdot\boldsymbol{\psi}_1 + \boldsymbol{m}_2\cdot(T_0\nabla s_0 - \boldsymbol{\zeta}_0) - T_0 Q_2 - T_1 Q_1 - T_2 Q_0 = 0,$$

or

$$\frac{\partial E_2}{\partial t}+(H_0-T_0 s_0) C_2+T_0\frac{\partial}{\partial t}(\rho_0 s_2+\rho_1 s_1+\rho_2 s_0)+T_0 s_0 \nabla\cdot \boldsymbol{m}_2$$

$$+\boldsymbol{m}_0\cdot(\boldsymbol{L}_2+T_0\nabla s_2+T_1\nabla s_1+T_2\nabla s_0-\boldsymbol{\zeta}_2-\boldsymbol{\psi}_2)+\nabla\cdot(\boldsymbol{m}_1 H_1)$$
$$-\boldsymbol{m}_1\cdot\boldsymbol{\psi}_1+\boldsymbol{m}_2\cdot(T_0\nabla s_0-\boldsymbol{\zeta}_0)-T_0 Q_2-T_1 Q_1-T_2 Q_0=0. \quad (27)$$

After use of (5c), this can be written

$$\frac{\partial E_2}{\partial t}+(H_0-T_0 s_0) C_2+\boldsymbol{m}_0\cdot(\boldsymbol{L}_2-\boldsymbol{\psi}_2)+T_0(S_2-Q_2)+\nabla\cdot(\boldsymbol{m}_1 H_1)-\boldsymbol{m}_1\cdot\boldsymbol{\psi}_1$$
$$+T_1\nabla\cdot(\boldsymbol{m}_0 s_1)-T_0\nabla\cdot(\boldsymbol{m}_1 s_1)-\boldsymbol{m}_0\cdot\boldsymbol{\zeta}_2-\boldsymbol{m}_2\cdot\boldsymbol{\zeta}_0-T_1 Q_1=0, \quad (28)$$

and in this case it is seen that (9) do not annihilate every term in the general energy equation for $n=2$. If it is noted that $\boldsymbol{\zeta}_2=\boldsymbol{\xi}_0\times\boldsymbol{u}_2+\boldsymbol{\xi}_1\times\boldsymbol{u}_1+\boldsymbol{\xi}_2\times\boldsymbol{u}_0$ and $\boldsymbol{m}_2=\rho_0\boldsymbol{u}_2+\rho_1\boldsymbol{u}_1+\rho_2\boldsymbol{u}_0$, then it can be shown that the vorticity terms in (28) reduce to $-\rho_0\boldsymbol{u}_0\cdot(\boldsymbol{\xi}_1\times\boldsymbol{u}_1)-\rho_1\boldsymbol{u}_1\cdot(\boldsymbol{\xi}_0\times\boldsymbol{u}_0)$. Hence, after (9) are used, what remains in (28) is

$$\frac{\partial E_2}{\partial t}+\nabla\cdot[\boldsymbol{m}_1(H_1-T_0 s_1)+\boldsymbol{m}_0 T_1 s_1]-\boldsymbol{m}_1\cdot\boldsymbol{\psi}_1-T_1 Q_1$$
$$+\boldsymbol{m}_1 s_1\cdot\nabla T_0-\boldsymbol{m}_0 s_1\cdot\nabla T_1-\rho_0\boldsymbol{u}_0\cdot(\boldsymbol{\xi}_1\times\boldsymbol{u}_1)-\rho_1\boldsymbol{u}_1\cdot(\boldsymbol{\xi}_0\times\boldsymbol{u}_0)=0. \quad (29)$$

Reintroduction of the definitions of $\boldsymbol{\psi}$ and Q yields

$$\boldsymbol{m}_1\cdot\boldsymbol{\psi}_1=m_{1_j}\left(\frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i}\right)_1$$
$$=\frac{\partial}{\partial x_i}\left[m_{1_j}\left(\frac{P_{ij}}{\rho}\right)_1\right]-\left(\frac{P_{ij}}{\rho}\right)_1\frac{\partial m_{1_j}}{\partial x_i}+m_{1_j}\left(\frac{P_{ij}}{\rho^2}\frac{\partial\rho}{\partial x_i}\right)_1, \quad (30)$$

and

$$T_1 Q_1=T_1\left(\frac{\Phi-\nabla\cdot\boldsymbol{q}}{T}\right)_1$$
$$=T_1\left(\frac{\Phi}{T}\right)_1-\nabla\cdot\left[T_1\left(\frac{\boldsymbol{q}}{T}\right)_1\right]+\left(\frac{\boldsymbol{q}}{T}\right)_1\cdot\nabla T_1-T_1\left(\frac{\boldsymbol{q}\cdot\nabla T}{T^2}\right)_1, \quad (31)$$

in which, as usual, the subscript 1 denotes the first-order term of the corresponding quantity expanded according to (4). Equation (29) then assumes the form

$$\frac{\partial E_2}{\partial t}+\frac{\partial W_{2i}}{\partial x_i}=-D_2, \quad (32)$$

with E_2 given by (26), and with

$$W_{2_i}=m_{1_i}(H_1-T_0 s_1)-m_{1_j}\left(\frac{P_{ij}}{\rho}\right)_1+m_{0_i}T_1 s_1+T_1\left(\frac{q_i}{T}\right)_1; \quad (33)$$

$$D_2=-\rho_0\boldsymbol{u}_0\cdot(\boldsymbol{\xi}_1\times\boldsymbol{u}_1)-\rho_1\boldsymbol{u}_1\cdot(\boldsymbol{\xi}_0\times\boldsymbol{u}_0)+s_1\boldsymbol{m}_1\cdot\nabla T_0-s_1\boldsymbol{m}_0\cdot\nabla T_1+\left(\frac{P_{ij}}{\rho}\right)_1\frac{\partial m_{1_j}}{\partial x_i}$$
$$-m_{1_j}\left(\frac{P_{ij}}{\rho^2}\frac{\partial\rho}{\partial x_i}\right)_1-T_1\left(\frac{\Phi}{T}\right)_1-\left(\frac{\boldsymbol{q}}{T}\right)_1\cdot\nabla T_1+T_1\left(\frac{\boldsymbol{q}\cdot\nabla T}{T^2}\right)_1. \quad (34)$$

When it is noted that $H_1 - T_0 s_1 = h_1 + \boldsymbol{u}_0 \cdot \boldsymbol{u}_1 - T_0 s_1 = p_1/\rho_0 + \boldsymbol{u}_0 \cdot \boldsymbol{u}_1$ from (13), it is seen that the first term on the right-hand side of (33) is precisely the acoustic energy flux of (17). In addition, in the absence of first-order entropy fluctuations, E_2 of (26) is the same as the acoustic energy density E_a of (16). Thus, for homentropic, irrotational ideal fluid flow, (32) reduces to (15).

In fact, the result embodied in (32), (26), (33) and (34) is the most general energy corollary of its type applicable to the system (6). Just as for the various special cases discussed by earlier authors, it can be derived by straightforward algebraic manipulation of that system. To those who would argue that such a derivation is more direct than the formal perturbation employed here, the author would point out that the algebraic effort required to obtain the result in its full generality is actually about the same by either method. One advantage of the current approach, however, is that it provides general answers to the questions posed early in this paper about the meaning of such corollaries. Equation (32) contains only first-order perturbation quantities, and yet, because of (28), it is a *complete*, consistent representation of the principle of total fluid energy conservation at order δ^2; the conservation law on total energy is affected by the solution of the second-order system (7) only at orders δ^3 and higher, even in the most general of circumstances.

Although the primary purpose for obtaining (32) here has been to illustrate its meaning relative to (18) and to provide motivation for the generalization to be developed later, this result itself is one which seems not to have appeared elsewhere. As a result, it merits some discussion, which will be kept relatively brief in view of current purposes. As is customary, the quantity E_2 of (26) is defined here as the first-order disturbance energy density, and $\boldsymbol{W}_2$ is the corresponding first-order disturbance energy flux vector. The source term D_2, according to (32), then represents the rate per unit volume at which first-order disturbance energy is being dissipated. Equation (32) indicates that this dissipation arises not only from thermoviscous effects but also from transfers of disturbance energy between the steady velocity field and the disturbance vorticity and between the steady vorticity and the disturbance velocity. In addition, D_2 contains an effective thermal contribution associated with the first-order entropy field. The flux vector $\boldsymbol{W}_2$ consists of (17) augmented by a contribution from the rate of work of viscous stresses and by an explicit heat flux term and an effective thermal component arising from the flux of first-order entropy. It is emphasized, however, that complexities introduced by the presence of the various denominators in (33) and (34) seem to preclude a strict physical interpretation of each separate term that ultimately appears in (32). For example, the viscous stress power term in $\boldsymbol{W}_2$ is

$$m_{1_j}\left(\frac{P_{ij}}{\rho}\right)_1 = (\rho_0 u_{1_j} + \rho_1 u_{0_j})\left(\frac{P_{1_{ij}}}{\rho_0} - \frac{\rho_1 P_{0_{ij}}}{\rho_0^2}\right), \tag{35}$$

with similarly complicated expressions resulting from the six other such terms in (33) and (34). As for the disturbance energy density of (26), it is seen to include the usual 'potential' energy per unit volume resulting from reversible elastic compression, $p_1^2/2\rho_0 c_0^2$, as well as the first-order measure of disturbance kinetic energy per unit volume and a contribution, discussed by Pierce (1981) for the case $\boldsymbol{u}_0 = 0$, proportional to the square of the first-order entropy.

Besides being of inherent value in leading to physical understanding of the mechanisms of energy transport associated with disturbances in moving, non-uniform media, (32) can be useful for identification of general conditions under which known simpler results are valid. For example, (15) is known to hold for homentropic, irrotational ideal fluid flow. In fact, for an ideal fluid, (26) and (33) reduce to (16) and (17) provided only that s_1 vanishes, and $D_2 = 0$ in this case if $\boldsymbol{\xi} = 0$. Thus, (15) is valid for irrotational ideal flow even in the presence of entropy gradients in the basic steady flow so long as first-order entropy fluctuations are absent.

Another known result follows from (32) if one considers a Newtonian viscous fluid for which

$$P_{ij} = -\tfrac{2}{3}\mu\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right). \tag{36}$$

Here the customary Stokes' hypothesis has been invoked, and μ is the dynamic viscosity. Let the heat flux be according to Fourier's law, $q_j = -K\,\partial T/\partial x_j$, where K is the thermal conductivity, assumed constant here. If a quiescent basic state $\boldsymbol{u}_0 = 0$ is considered, it follows that $P_{0_{ij}} = 0$ and that $\Phi_0 = \Phi_1 = 0$. Then (33) reduces to

$$W_{2_i} = p_1 u_{1_i} - u_{1_j} P_{1_{ij}} - \frac{KT_1}{T_0}\left(\frac{\partial T_1}{\partial x_i} - \frac{T_1}{T_0}\frac{\partial T_0}{\partial x_i}\right), \tag{37}$$

and (34) becomes

$$D_2 = s_1 \boldsymbol{m}_1 \cdot \nabla T_0 + \frac{P_{1_{ij}}}{\rho_0}\frac{\partial}{\partial x_i}(\rho_0 u_{1_j}) - \rho_0 u_{1_j}\left(\frac{P_{1_{ij}}}{\rho_0^2}\frac{\partial \rho_0}{\partial x_i}\right)$$
$$+ \frac{K}{T_0}\left(\nabla T_1 - \frac{T_1}{T_0}\nabla T_0\right)\cdot\nabla T_1 - \frac{2KT_1}{T_0^2}\left[\nabla T_0 \cdot \nabla T_1 - \frac{T_1}{T_0}(\nabla T_0)^2\right],$$

or

$$D_2 = s_1 \boldsymbol{m}_1 \cdot \nabla T_0 + P_{1_{ij}}\frac{\partial u_{1_j}}{\partial x_i} + \frac{K}{T_0}\left[(\nabla T_1)^2 - 3\frac{T_1}{T_0}\nabla T_0 \cdot \nabla T_1 + 2\frac{T_1^2}{T_0^2}(\nabla T_0)^2\right]. \tag{38}$$

It is easily shown using (36) that if $\nabla T_0 = 0$, (37) and (38) are precisely the expressions given by Pierce (1981, equations 10.2.5*b*, *c*), who assumed a completely uniform ambient state. The current analysis provides the generalization of Pierce's viscous energy corollary to the case in which temperature (entropy) gradients are present in the stationary basic state.

Equation (32) is also directly related to the acoustic result of Morfey (1971). Morfey was concerned with constructing an acoustic energy law having the density and flux given by (16) and (17) with ρ_1 replaced by p_1/c_0^2. To accomplish this he split the velocity vector $\boldsymbol{u}$ into irrotational and solenoidal parts and included all thermoviscous terms and all terms containing the vortical velocity, vorticity and entropy fluctuations in a source term on the right-hand side of (15). If this process is applied to (32) it can be shown, after considerable algebra, that Morfey's result follows. In general, however, viscous stress power and heat flux terms properly belong in an energy flux vector and, as Pierce (1981) indicates, the measure of disturbance energy density should include the s_1^2 term of (26). In any event, the above approach is not pursued further here because the main objective is to develop, in the next section, an exact corollary valid even for a nonlinear disturbance theory in which the acoustic–vortical split of velocity cannot be unambiguously achieved.

Finally, it is noted that Möhring (1973) followed quite a different approach to the derivation of an energy corollary for the linear system (6). He considered the ideal fluid case and sought a conservation law in which the remaining terms involving entropy and vorticity in the source expression (34) were included on the left-hand side of (32). This was accomplished, but at the expense of introducing an energy density and flux expressed in terms of Clebsch potentials, which are not uniquely defined. As a result, the density is not unique, and the flux is unique only on a time-averaged basis. While a source-free energy corollary would be preferable to one like (32), it appears that Möhring's ideal fluid corollary is practically useful mainly for studies of time-averaged energy transport.

4. Exact energy corollary

The formal approach taken in the preceding section has shown that, at the first three orders, the nth-order general equation assumes the form of a sum of certain multiples of basic steady flow quantities with the continuity, linear momentum and entropy equations at order n as well as, for $n = 2$, terms constituting the first-order disturbance energy corollary, which involves only q_{n-1} perturbation quantities. This suggests that it is of interest to explore the possibility that this pattern repeats itself at all orders. What will be done here is to write the exact energy equation (18) and to subtract from it the same multiples of the exact equations (8) as occurred in nth-order form in (21), (24) and (28). Because the terms in (21) and (24) vanished identically, it can at least be expected that the new exact equation will be devoid of zero- and first-order terms when expanded according to the form (4). In fact, as will be seen shortly, the new equation assumes the form of an exact energy corollary analogous to (32) whose leading-order representation under the scheme (4) is at $O(\delta^2)$ and is identical to (32).

According to the procedure suggested above, the sum of $(H_0 - T_0 s_0)$ times equation ($8a$), T_0 times equation ($8c$), and the scalar product of $\boldsymbol{m}_0$ with equation ($8b$) are subtracted from the general energy equation ($18b$). This yields

$$\frac{\partial}{\partial t}[\rho H - p - \rho(H_0 - T_0 s_0) - \rho T_0 s - \boldsymbol{m}_0\cdot\boldsymbol{u}] + \nabla\cdot(\boldsymbol{m}H) - \boldsymbol{m}\cdot\boldsymbol{\psi} - TQ - (H_0 - T_0 s_0)\,\nabla\cdot\boldsymbol{m}$$
$$- T_0\nabla\cdot(\boldsymbol{m}s) + T_0 Q - \boldsymbol{m}_0\cdot[\boldsymbol{\zeta} + \nabla H - T\nabla s - \boldsymbol{\psi}] = 0, \quad (39)$$

which can be rearranged, using (5), into

$$\frac{\partial}{\partial t}[\rho(H-H_0) - \rho T_0(s-s_0) - \boldsymbol{m}_0\cdot\boldsymbol{u} - p] + \nabla\cdot[\boldsymbol{m}H - \boldsymbol{m}(H_0 - T_0 s_0) - \boldsymbol{m}sT_0 - \boldsymbol{m}_0 H + \boldsymbol{m}_0 Ts]$$
$$-(\boldsymbol{m}-\boldsymbol{m}_0)\cdot\boldsymbol{\psi} - (T-T_0)\,Q + \boldsymbol{m}\cdot(\boldsymbol{\psi}_0 - \boldsymbol{\zeta}_0) + (s-s_0)\,\boldsymbol{m}\cdot\nabla T_0 - \boldsymbol{m}_0\cdot\boldsymbol{\zeta} - s\boldsymbol{m}_0\cdot\nabla T = 0. \quad (40)$$

Now consider the divergence term in (40); it can be rewritten as

$$\nabla\cdot[(\boldsymbol{m}-\boldsymbol{m}_0)(H-H_0) - \boldsymbol{m}T_0(s-s_0) + \boldsymbol{m}_0 T(s-s_0) - \boldsymbol{m}_0 H_0 + \boldsymbol{m}_0 Ts_0]. \quad (41)$$

From (5) (or $19a$), it follows that

$$\nabla\cdot(\boldsymbol{m}_0 H_0) = \boldsymbol{m}_0\cdot\boldsymbol{\psi}_0 + T_0 Q_0,$$

and, from ($5c$), that

$$\nabla\cdot(\boldsymbol{m}_0 Ts_0) = s_0\,\boldsymbol{m}_0\cdot\nabla T + TQ_0,$$

which allow (41) to be put in the form

$$\nabla\cdot[(\boldsymbol{m}-\boldsymbol{m}_0)(H-H_0) - \boldsymbol{m}T_0(s-s_0) + \boldsymbol{m}_0 T(s-s_0)]$$
$$- \boldsymbol{m}_0\cdot\boldsymbol{\psi}_0 + s_0\,\boldsymbol{m}_0\cdot\nabla T + (T-T_0)\,Q_0. \quad (42)$$

Substitution of this back into (40), and use of the fact that $\boldsymbol{u}_0$ and p_0 are independent of t, then enables that result to be written as

$$\frac{\partial}{\partial t}\{\rho[H-H_0-T_0(s-s_0)]-\boldsymbol{m}_0\cdot(\boldsymbol{u}-\boldsymbol{u}_0)-(p-p_0)\}$$
$$+\nabla\cdot\{(\boldsymbol{m}-\boldsymbol{m}_0)[H-H_0-T_0(s-s_0)]+\boldsymbol{m}_0(T-T_0)(s-s_0)\}-(\boldsymbol{m}-\boldsymbol{m}_0)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_0)$$
$$-(T-T_0)(Q-Q_0)-\boldsymbol{m}_0\cdot\boldsymbol{\zeta}-\boldsymbol{m}\cdot\boldsymbol{\zeta}_0+(s-s_0)\boldsymbol{m}\cdot\nabla T_0-(s-s_0)\boldsymbol{m}_0\cdot\nabla T=0. \quad (43)$$

Upon introducing the definitions of $\boldsymbol{\psi}$ and Q, it is seen that

$$(\boldsymbol{m}-\boldsymbol{m}_0)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}_0)=(m_j-m_{0_j})\left(\frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i}-\frac{1}{\rho_0}\frac{\partial P_{0_{ij}}}{\partial x_i}\right)$$
$$=\frac{\partial}{\partial x_i}\left[(m_j-m_{0_j})\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)\right]-\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)\frac{\partial}{\partial x_i}(m_j-m_{0_j})$$
$$+(m_j-m_{0_j})\left(\frac{P_{ij}}{\rho^2}\frac{\partial\rho}{\partial x_i}-\frac{P_{0_{ij}}}{\rho_0^2}\frac{\partial\rho_0}{\partial x_i}\right), \quad (44)$$

and that

$$(T-T_0)(Q-Q_0)=(T-T_0)\left(\frac{\Phi}{T}-\frac{\Phi_0}{T_0}-\frac{\nabla\cdot\boldsymbol{q}}{T}+\frac{\nabla\cdot\boldsymbol{q}_0}{T_0}\right)$$
$$=(T-T_0)\left(\frac{\Phi}{T}-\frac{\Phi_0}{T_0}\right)-\nabla\cdot\left[(T-T_0)\left(\frac{\boldsymbol{q}}{T}-\frac{\boldsymbol{q}_0}{T_0}\right)\right]$$
$$+\left(\frac{\boldsymbol{q}}{T}-\frac{\boldsymbol{q}_0}{T_0}\right)\cdot\nabla(T-T_0)-(T-T_0)\left(\frac{\boldsymbol{q}\cdot\nabla T}{T^2}-\frac{\boldsymbol{q}_0\cdot\nabla T_0}{T_0^2}\right). \quad (45)$$

Then, after substituting (44) and (45) into (43) and noting the fact that $\boldsymbol{m}\cdot\boldsymbol{\zeta}=\rho\boldsymbol{u}\cdot(\boldsymbol{\xi}\times\boldsymbol{u})\equiv 0$ and $\boldsymbol{m}_0\cdot\boldsymbol{\zeta}_0\equiv 0$, it follows that (43) assumes the form

$$\frac{\partial E}{\partial t}+\frac{\partial W_i}{\partial x_i}=-D, \quad (46)$$

in which

$$E=\rho[H-H_0-T_0(s-s_0)]-\boldsymbol{m}_0\cdot(\boldsymbol{u}-\boldsymbol{u}_0)-(p-p_0); \quad (47)$$

$$W_i=(m_i-m_{0_i})[H-H_0-T_0(s-s_0)]+m_{0_i}(T-T_0)(s-s_0)$$
$$-(m_j-m_{0_j})\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)+(T-T_0)\left(\frac{q_i}{T}-\frac{q_{0_i}}{T_0}\right); \quad (48)$$

$$D=(\boldsymbol{m}-\boldsymbol{m}_0)\cdot[\boldsymbol{\xi}\times\boldsymbol{u}-\boldsymbol{\xi}_0\times\boldsymbol{u}_0+(s-s_0)\nabla T_0]-(s-s_0)\boldsymbol{m}_0\cdot\nabla(T-T_0)$$
$$+\left(\frac{P_{ij}}{\rho}-\frac{P_{0_{ij}}}{\rho_0}\right)\frac{\partial}{\partial x_i}(m_j-m_{0_j})-(m_j-m_{0_j})\left(\frac{P_{ij}}{\rho^2}\frac{\partial\rho}{\partial x_i}-\frac{P_{0_{ij}}}{\rho_0^2}\frac{\partial\rho_0}{\partial x_i}\right)$$
$$-(T-T_0)\left(\frac{\Phi}{T}-\frac{\Phi_0}{T_0}\right)-\left(\frac{\boldsymbol{q}}{T}-\frac{\boldsymbol{q}_0}{T_0}\right)\cdot\nabla(T-T_0)+(T-T_0)\left(\frac{\boldsymbol{q}\cdot\nabla T}{T^2}-\frac{\boldsymbol{q}_0\cdot\nabla T_0}{T_0^2}\right). \quad (49)$$

Before entering into any discussion of (46)–(49), which are the major result of the current paper, it is noted that the source termed defined by (49) is expressed above in a form which allows easy term-by-term comparison with (34). It can be written in somewhat simpler form. The terms containing the heat flux vector are

$$-\frac{\boldsymbol{q}}{T}\cdot\left[\nabla(T-T_0)-\left(1-\frac{T_0}{T}\right)\nabla T\right]+\frac{\boldsymbol{q}_0}{T_0}\cdot\left[\nabla(T-T_0)-\left(\frac{T}{T_0}-1\right)\nabla T_0\right]$$
$$=-\left(\frac{T_0\boldsymbol{q}}{T}-\frac{T\boldsymbol{q}_0}{T_0}\right)\cdot\left(\frac{\nabla T}{T}-\frac{\nabla T_0}{T_0}\right), \quad (50)$$

and the contribution of the viscous stress components to D is equivalent to

$$P_{ij}\frac{\partial}{\partial x_i}\left(\frac{m_j-m_{0_j}}{\rho}\right)-P_{0_{ij}}\frac{\partial}{\partial x_i}\left(\frac{m_j-m_{0_j}}{\rho_0}\right)=\Phi+\Phi_0-P_{ij}\frac{\partial}{\partial x_i}\left(\frac{m_{0_j}}{\rho}\right)-P_{0_{ij}}\frac{\partial}{\partial x_i}\left(\frac{m_j}{\rho_0}\right)$$
$$=\left(1-\frac{\rho}{\rho_0}\right)\Phi+\left(1-\frac{\rho_0}{\rho}\right)\Phi_0+(P_{ij}-P_{0_{ij}})\frac{\partial}{\partial x_i}\left(\frac{m_j}{\rho_0}-\frac{m_{0_j}}{\rho}\right)$$
$$-\left(\frac{m_jP_{ij}}{\rho_0}-\frac{m_{0_j}P_{0_{ij}}}{\rho}\right)\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x_i}-\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x_i}\right). \quad (51)$$

Use of (50) and (51) in (49) then leads to an alternative expression for the source term of the form

$$D=(\boldsymbol{m}-\boldsymbol{m}_0)\cdot[\boldsymbol{\xi}\times\boldsymbol{u}-\boldsymbol{\xi}_0\times\boldsymbol{u}_0+(s-s_0)\nabla T_0]-(s-s_0)\boldsymbol{m}_0\cdot\nabla(T-T_0)$$
$$-(\rho T-\rho_0T_0)\left(\frac{\Phi}{\rho_0T}-\frac{\Phi_0}{\rho T_0}\right)$$
$$+(P_{ij}-P_{0_{ij}})\frac{\partial}{\partial x_i}\left(\frac{m_j}{\rho_0}-\frac{m_{0_j}}{\rho}\right)-\left(\frac{m_jP_{ij}}{\rho_0}-\frac{m_{0_j}P_{0_{ij}}}{\rho}\right)\frac{\partial}{\partial x_i}\ln\left(\frac{\rho}{\rho_0}\right)$$
$$-\left(\frac{T_0\boldsymbol{q}}{T}-\frac{T\boldsymbol{q}_0}{T_0}\right)\cdot\nabla\ln\left(\frac{T}{T_0}\right). \quad (52)$$

The above result, (46)–(49), is, as noted, the energy corollary which is the primary goal of the current work. It is emphasized that it is an *exact* relation satisfied by arbitrary disturbances to a completely general steady viscous flow. No approximations or expansions of the fluid quantities have been used in deriving it; most importantly, it has not been assumed that the disturbance quantities are governed by the linear system (6). The quantity E will be termed here the disturbance energy density, with $\boldsymbol{W}$ the corresponding disturbance energy flux vector. The relationship of the disturbance energy quantities to their first-order counterparts of (32) and to the total energy density and flux of (18) will be discussed in the following section.

5. Discussion and simple application

Clearly, the definition of a pair of quantities like E and $\boldsymbol{W}$ so that they satisfy a conservation equation of the form (46) is, to a large extent, arbitrary. For example, any linear combination of conservation equations having consistent dimensions is itself another one. Equation (46), however, has a particular significance among other such equations that could be derived because expansion of E, $\boldsymbol{W}$ and D according to (4) yields no terms of $O(1)$ or $O(\delta)$ and yields δ^2 coefficients which contain only first-order perturbation quantities q_1. This property of the corollary (46) is what would render it useful regardless of the particular first-order expressions that constitute its δ^2 coefficients, but its construction in view of the results of §3 ensures in addition that they are the same as in (26), (33) and (34). Therefore, at leading order in the perturbation scheme (4), equation (46) is precisely the first-order corollary of (32). The quantities E and $\boldsymbol{W}$ thus provide an answer to the fundamental question posed in the final paragraph of §1 of this paper in that they represent *exact* fluid dynamic quantities which are approximated at lowest order in the scheme (4) by the various commonly employed measures of energy density and flux. In homentropic ideal flow, for example, the disturbance energy flux is simply equal to the product of the disturbance in the mass flux and the disturbance in the stagnation enthalpy, and this quantity is approximated in linearized acoustic theory by the acoustic energy flux (17).

Explicit demonstration of the properties mentioned above is achieved by expanding (47)–(49) according to (4). From (47),

$$E = \rho(e+\tfrac{1}{2}u^2-h_0-\tfrac{1}{2}u_0^2)+p_0-\rho T_0(s-s_0)-\rho_0\boldsymbol{u}_0\cdot(\boldsymbol{u}-\boldsymbol{u}_0),$$

and, by construction, $E_0 \equiv 0$. Its first-order representation is

$$(\rho e)_1+\tfrac{1}{2}\rho_1 u_0^2+\rho_0\boldsymbol{u}_0\cdot\boldsymbol{u}_1-\rho_1(h_0+\tfrac{1}{2}u_0^2)-\rho_0 T_0 s_1-\rho_0\boldsymbol{u}_0\cdot\boldsymbol{u}_1,$$

and, because of (12), this also vanishes identically. At order δ^2 E is

$$(\rho e)_2+\tfrac{1}{2}\rho_0 u_1^2+\rho_0\boldsymbol{u}_0\cdot\boldsymbol{u}_2+\rho_1\boldsymbol{u}_0\cdot\boldsymbol{u}_1+\tfrac{1}{2}\rho_2 u_0^2 \\ -\rho_2(h_0+\tfrac{1}{2}u_0^2)-\rho_1 T_0 s_1-\rho_0 T_0 s_2-\rho_0\boldsymbol{u}_0\cdot\boldsymbol{u}_2,$$

which, after (12) is used for $(\rho e)_2$, is found to reduce exactly to E_2 as given by (26). Hence, $E = \delta^2 E_2+\dots$ when the perturbation scheme (4) is employed.

The flux $\boldsymbol{W}$ and the source expression D exhibit the same behaviour. In this case, because they are written in (48) and (49) entirely in terms of products of disturbances, it is obvious that $\boldsymbol{W}_0$, $\boldsymbol{W}_1$, D_0 and D_1 all vanish. For the same reason, their representations at order δ^2 will contain only products of first-order disturbance quantities, and it can be seen by direct comparison of (48) and (33) and of (49) and (34) that they will be exactly $\boldsymbol{W}_2$ and D_2. It follows, therefore, that the leading-order representation of the exact corollary (46) under the scheme (4) is precisely (32).

It is important to note that, as is the case in all previous analyses of the current type, neither E nor $\boldsymbol{W}$ are equal to the changes in the respective total fluid energy density E_{TOT} and flux $\boldsymbol{W}_{\text{TOT}}$ owing to the existence of the disturbance. For example, consider the ideal fluid case; from (18), $\Delta\boldsymbol{W}_{\text{TOT}} = \boldsymbol{m}H-\boldsymbol{m}_0 H_0$ so that

$$\boldsymbol{W} = \Delta\boldsymbol{W}_{\text{TOT}}-(\boldsymbol{m}-\boldsymbol{m}_0)\,[H-T_0(s-s_0)]-\boldsymbol{m}_0[H-H_0-(T-T_0)\,(s-s_0)]. \tag{53}$$

The additional terms subtracted in (53) remove from $\Delta\boldsymbol{W}_{\text{TOT}}$ all terms which are linear in the disturbance quantities. This is the reason why the second-order representation of $\boldsymbol{W}$ in (33) contains only first-order perturbation quantities. Similar remarks apply to E. It does not seem to be possible to give a simple physical interpretation of all of the additional subtracted terms, but it can be seen that they account for the portion of total energy flux not strictly associated with the mechanical and thermal processes of propagation of the disturbance through the basic flow. For example, the subtracted terms in (53) which contain the pressure are

$$(\rho\boldsymbol{u}-\rho_0\boldsymbol{u}_0)\frac{p_0}{\rho_0}+\rho_0\boldsymbol{u}_0\left(\frac{p}{\rho}-\frac{p_0}{\rho_0}\right) = p_0(\boldsymbol{u}-\boldsymbol{u}_0)+(p-p_0)\,\boldsymbol{u}_0+\text{QT},$$

where QT stands for terms quadratic in the disturbance quantities $p-p_0$, $\rho-\rho_0$ and $\mathbf{u}-\mathbf{u}_0$. This indicates that the pressure contribution subtracted from ΔW_{TOT} in (53) includes the rate of work done by the ambient pressure against the disturbance velocity and its counterpart, the rate of work done by the disturbance pressure against the ambient velocity. Similarly, the extra terms also include the simple convection of ambient total energy by the disturbance velocity and of disturbance total energy by the ambient velocity.

In view of its completely general form, (46) is probably of most value in conjunction with special cases for which some of its terms are absent. As indicated earlier, the author has applied a highly specialized version of (46) in a study of nonlinear quasi-one-dimensional waves propagating through a near-sonic throat flow. Details of that work confirm the fact that the general energy corollary leads to a consistent accounting for the transport of disturbance energy in a situation where linearized theory is inadequate to describe the disturbance field. Here, as an example of the utility of (46), a much simpler nonlinear problem will be discussed, and that only in enough detail to illustrate the application of the corollary.

Consider plane wave propagation in an ideal gas. As is well known, even in the absence of explicit thermoviscous effects, shock waves which develop in such fields give rise to energy losses which attenuate waves as they travel. This is a situation in which (46) with $s-s_0 \neq 0$ can provide a convenient means of accounting for the losses. The 'piston' problem, in which data are specified on $x=0$ and plane waves move out into $x>0$, has been analysed in detail by Whitham (1974) for the case of finite-amplitude disturbances travelling into a uniform, quiescent state. So long as the data are such as to generate sufficiently weak shocks, a first approximate solution for the disturbance field is obtained by assuming that the flow is homentropic. In this case equations (1) are equivalent to

$$R_t^{\pm}+(u\pm c)R_x^{\pm}=0, \tag{54}$$

in which $R^{\pm}$ are the Riemann invariants,

$$R^{\pm}=\frac{2c}{\gamma-1}\pm u, \tag{55}$$

and the subscript notation for derivatives is employed. The pressure and density in the flow are determined from

$$\frac{c^2}{c_0^2}=\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}, \quad p=\frac{\rho_0 c_0^2}{\gamma}\left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{56a, b}$$

where γ is the specific heat ratio and the subscript 0 again denotes the ambient state.

For the current purposes, however, a somewhat different problem will be considered. Assume that the disturbance propagates upstream into a gas which is in uniform motion at speed U in the negative x-direction. Define a dimensionless velocity perturbation μ and sound speed perturbation α according to

$$u=U(-1+\mu), \quad c=c_0(1+\alpha), \tag{57a, b}$$

and assume that the basic steady flow Mach number $M=U/c_0$ is near unity. This introduces a small parameter $\epsilon=1-M$ into the problem, and it becomes amenable to a formal perturbation analysis which describes waves of small amplitude propagating into an oncoming near-sonic flow. The details of this analysis will be omitted here, but it is noted that such an approach is of value because it leads to an analytical determination of a number of results associated with the development of shocks in the disturbance quantities and with the ultimate non-simple wave nature and non-isentropy of the solution. The problem will be discussed in detail in a subsequent paper.

In order to avoid extensive algebra here, however, it will again be anticipated that the leading approximation will be homentropic so that (54) and (55) are sufficient. After using (57*a*, *b*), they can be written

$$R^{\pm}_{T}+[M(-1+\mu)\pm(1+\alpha)]R^{\pm}_{x}=0, \tag{58}$$

$$R^{\pm}=c_0\left[\frac{2}{\gamma-1}(1+\alpha)\pm M(-1+\mu)\right], \tag{59}$$

in which $T=c_0 t$. If it is further assumed that the appropriate solution of (58) is a simple wave, then R^- has the constant value $2c_0/(\gamma-1)+U$ so that

$$\alpha=\tfrac{1}{2}(\gamma-1)M\mu=\tfrac{1}{2}(\gamma-1)(1-\epsilon)\mu, \tag{60}$$

and (58) reduces to

$$\mu_T+[\epsilon+\tfrac{1}{2}(\gamma+1)\mu]\mu_x=0. \tag{61}$$

To solve (61) for $\epsilon \ll 1$ requires no involved perturbation technique; one simply notes that if μ and μ_T are $O(\epsilon)$ then μ_x is $O(1/\epsilon)$ uniformly in x. Thus it is sufficient to introduce a 'fast' spatial variable $X=x/\epsilon$ and to seek $\mu=f(X,T)$ as a regular perturbation in powers of ϵ. Accordingly, let

$$\mu(x,T;\epsilon)=\epsilon\mu_1(X,T)+\epsilon^2\mu_2(X,T)+\ldots, \tag{62}$$

which, upon substitution into (61), yields a sequence of equations governing the μ_n. The leading term satisfies

$$\mu_{1_T}+[1+\tfrac{1}{2}(\gamma+1)\mu_1]\mu_{1_X}=0. \tag{63}$$

Let μ be specified at $x=0:\mu(0,T;\epsilon)=\epsilon\phi(T)$. Then the solution of equation (63) satisfies $\mu_1(0,t)=\phi(T)$ and it can be written in parametric form as

$$\mu_1(X,T)=\phi(\tau);\quad X=[1+\tfrac{1}{2}(\gamma+1)\phi(\tau)](T-\tau), \tag{64}$$

where the characteristic parameter $\tau(X,T)$ is chosen so that $\tau(0,T)\equiv T$. In fact, $\epsilon\mu_1$ is the exact continuous solution of (61) for arbitrary ϵ. When shocks occur, however, the solution is not a simple wave beyond $O(\epsilon)$, and $\mu_n \neq 0$ for $n>1$.

Further discussion of the above solution and its extension to higher order will not be presented here; the aim is simply to illustrate the application of the corollary (46) to the problem. For this case $H=c^2/(\gamma-1)+\tfrac{1}{2}u^2$, and the single component of the disturbance energy flux vector (48) is

$$W=(\rho u+\rho_0 U)\left(\frac{c^2-c_0^2}{\gamma-1}+\frac{u^2-U^2}{2}\right). \tag{65}$$

To obtain the consistent representation of W requires only that it be expanded according to the scheme (62). The second factor in (65) is written as

$$c_0^2\left[\frac{(1+\alpha)^2-1}{\gamma-1}+(1-\epsilon)^2\frac{(-1+\mu)^2-1}{2}\right].$$

Then expansion of (60) results in

$$\alpha_1=\tfrac{1}{2}(\gamma-1)\mu_1,\quad \alpha_2=\tfrac{1}{2}(\gamma-1)(\mu_2-\mu_1), \tag{66}$$

which, when substituted into the above factor, yields

$$\frac{c^2-c_0^2}{\gamma-1}+\tfrac{1}{2}(u^2-U^2) = \epsilon^2 c_0^2[\mu_1+\tfrac{1}{4}(\gamma+1)\mu_1^2]+\dots . \tag{67}$$

Similarly, the first factor in W is

$$\rho_0 c_0(1-\epsilon)\,[(1+\alpha)^{2/\gamma-1}(-1+\mu)+1]$$

after ($56a$) is used. Expansion of this according to (62) and (66) then gives

$$\rho u+\rho_0 U = \epsilon^2\rho_0 c_0[\mu_1+\tfrac{1}{4}(\gamma+1)\mu_1^2]+\dots . \tag{68}$$

Hence it follows that

$$W = \epsilon^4\rho_0 c_0^3(\mu_1+\tfrac{1}{4}(\gamma+1)\mu_1^2)^2+O(\epsilon^5) \tag{69}$$

for the current example, where μ_1 is given by (64).

Now consider the disturbance energy density; for the example here it is

$$E = \rho\left(\frac{c^2-c_0^2}{\gamma-1}+\tfrac{1}{2}(u^2-U^2)\right)+\rho_0 U(u+U)-(p-p_0). \tag{70}$$

If this is written in terms of α, μ and ϵ it can be shown, after some algebra, that it has the form

$$E = \epsilon^3\rho_0 c_0^2\mu_1(\mu_1+\tfrac{1}{6}(\gamma+1)\mu_1^2)+O(\epsilon^4). \tag{71}$$

If the expressions on the right-hand sides of (69) and (71) are written as $\epsilon^4 W_4$ and $\epsilon^3 E_3$, respectively, it follows that the energy corollary for the current example ($D =$ 0 in this case) is, at leading order,

$$\frac{\partial E_3}{\partial t}+\epsilon\frac{\partial W_4}{\partial x} = \frac{1}{c_0}\frac{\partial E_3}{\partial T}+\frac{\partial W_4}{\partial X} = 0. \tag{72}$$

It is easily seen that this conservation law also follows directly if (63) is multiplied by $2\mu_1[1+\tfrac{1}{4}(\gamma+1)\mu_1]$, which affirms the correctness of the result but does not shed much light on its physical meaning or on its relationship with the general energy equation.

Several points are worth noting in connection with (72). First, the leading terms in E and W contain only first-order perturbation quantities, as expected on the basis of the construction of the general corollary (46). However, this occurs here even though those terms do not appear until orders ϵ^3 and ϵ^4 in the singular perturbation scheme (62). Direct determination of the consistent first approximate form of (18) in this case would have required expansion of E_{TOT} and W_{TOT} up to orders ϵ^3 and ϵ^4, respectively, and utilization of the corresponding expansions of (8) up to the same orders in the singular scheme (62). This is an algebraic task of considerable magnitude, and it is avoided completely by the simple calculation of the leading approximations to (65) and (70). Second, the disturbance energy flux is one order smaller than the energy density. This is also expected, owing to the fact that spatial derivatives of W are one order larger than W itself in this case. Perhaps most important from a physical point of view is the conclusion which follows from the above that an $O(\epsilon)$ disturbance introduced into an oncoming flow at $M = 1-\epsilon$ imparts a disturbance energy to the flow whose density is only $O(\epsilon^3)$ and whose flux is only $O(\epsilon^4)$. Of course, for a *fixed* disturbance level and for M sufficiently greater than unity, no disturbance at all can be imparted to the flow. The result here indicates that this process of supersonic blocking takes place in a continuous manner for increasing M and begins when M is well below unity.

REFERENCES

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

BLOKHINTSEV, D. I. 1956 Acoustics of a nonhomogeneous moving medium. *Trans. NACA Tech. Mem.* 1399.

CANTRELL, R. H. & HART, R. W. 1964 Interaction between sound and flow in acoustic cavities: Mass, momentum and energy considerations. *J. Acoust. Soc. Am.* **36**, 697–706.

CHU, B. T. & APFEL, R. E. 1983 Are acoustic energy and potential energy density first- or second-order quantities? *Am. J. Phys.* **51**, 916–918.

ECKHART, C. 1948 Vortices and streams caused by sound waves. *Phys. Rev.* **73**, 68–76.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. Addison Wesley.

LIGHTHILL, M. J. 1978 *Waves in Fluids*. Cambridge University Press.

MÖHRING, W. 1973 On energy, group velocity and small damping of sound waves in ducts with shear flow. *J. Sound Vib.* **29**, 93–101.

MORFEY, C. L. 1971 Acoustic energy in non-uniform flows. *J. Sound Vib.* **14**, 159–169.

MYERS, M. K. 1981 Shock development in sound transmitted through a nearly choked flow. *AIAA 7th Aeroacoust. Conf. Paper* 81-2012.

MYERS, M. K. 1986*a* Generalization and extension of the law of acoustic energy conservation in a nonuniform flow. *AIAA 24th Aerospace Sci. Meeting, Paper* 86-0471.

MYERS, M. K. 1986*b* An exact energy corollary for homentropic flow. *J. Sound Vib.* **109**, 277–284.

MYERS, M. K. & CALLEGARI, A. J. 1982 Nonlinear theory of shocked sound propagation in a nearly choked duct flow. *NASA Contractor Rep.* CR 3549.

PIERCE, A. D. 1981 *Acoustics: An Introduction to its Physical Principles and Applications*. McGraw-Hill.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Wiley.